# Monetary Noether Theorem eliminates non-locality in monetary exchange

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Money is conserved only if the monetary system is homogeneous in time. Violation of this Noether relation leads to non-local interactions between money holders. Money is represented as a particle-antiparticle pair of assets and liabilities. Value conservation holds because transfers are invariant under exchange of trading roles. Money conservation however is systematically violated if monetary pair creation by credit is allowed. Normalization yields a fully symmetric monetary space: local, parallel transfers of assets and liabilities replace non-local, time-asymmetric credit operations. A liquidity market substitutes the credit market.

The main function of money is to store information on transactions in the economy and to establish a unit of account in the credit-debit record keeping of the banking system[1,2]. The symmetry properties of this economic memory are important because they entail the local conservation laws that are postulated in much of the literature on monetary theory and the statistical mechanics of money[3]. If a monetary space is not homogeneous in time, the money supply is not fixed, and non-local, unwarranted transfers occur. Consequently, precise storage of economic information is no longer guaranteed. Although these non-local interactions are known in economics as inflation tax, their microscopic effects have largely gone unnoticed in economics and econophysics: the mechanism by which transactions are recorded has economic consequences for all agents. Since the money supply in real world economies is largely driven by demand for credit, contemporary monetary systems systematically violate time homogeneity and consequently are rather imperfect stores of economic information[2].

The statistical mechanics approach has proven highly successful in determining the global properties of monetary systems subject to different boundary conditions[3,4,5,6]. All of this work however contains the local conservation of money as a postulate and deals only with nominal money, ignoring the effective redistributive taxation that comes with the creation of monetary units. In economics, the Friedman rule holds that a constant money supply improves welfare, however the problem of liquidity shortages under such a regime has not been addressed[7].

Symmetries of monetary spaces are therefore of high interest both in econophysics, because symmetries restrict system behavior, as well as economics, because such restrictions potentially lead to unnecessary inflexibility. Yet a monetary system homogeneous in time would be beneficial because it would keep the money supply constant.

In the following we show how invariance under time translation and exchange of trading roles leads to conserved quantities and present a fully symmetric toy model of a monetary system that addresses the problem of liquidity shortages through a new type of monetary transfer.

Economic theory treats money mainly as a means of exchange for goods and services. Such a purchase transfer is shown in Fig. 1a, where agent $i$ transfers an asset to $j$. The assets and liabilities of all other agents are unaffected by the transfer. We use a particle picture inspired by Feynman graphs to allow a time-ordered analysis[8]. The arrows represent

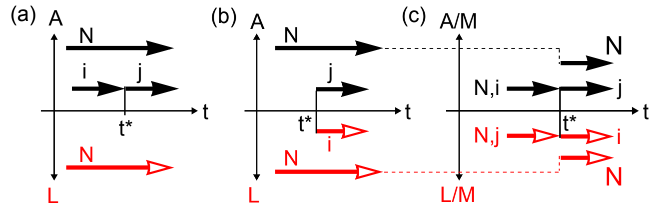


FIG. 1. Non-locality in monetary exchange. Assets A and liabilities L represented as particles moving in agent space $(N, i, j)$. (a) Bouncing (transfer) and (b) pair creation of monetary units are both valid ways for $i$ to pay $j$. (c) Normalization by money supply $M$ shows the effect of money creation on the relative monetary wealth: it constitutes indirect transfers of assets $(N, i) \to j$ and liabilities $(N, j) \to i$.

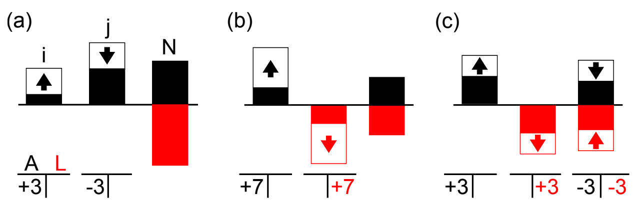


FIG. 2. Bookkeeping representation of FIG. 1. (a) An asset transfer does not change the total money supply $M$ and hence does not change the relative standing of other agents. (b) Money creation in bookkeeping appears to be neutral. (c) Normalization reveals indirect transfers; the white areas of $N$ add up to the gains of $i$ and $j$.

black A-particles and red L-particles moving along the time axis. Assets have a monetary value $p > 0$, liabilities $p < 0$. The corresponding bookkeeping picture that records the account changes in balance sheets is shown in FIG. 2a.

However, the transfer can also be executed by creation of credit, if agent $i$ takes out a loan at time $t^*$ to pay $j$. FIG. 1b and FIG. 2b show how in this case new money is created by the bank as a pair consisting of an asset 'particle' A (e.g. a bank deposit) and a 'liability' anti-particle L (e.g. a loan) in the bookkeeping of banks. Whenever a loan is taken out, new money comes into existence[2],[9]: loans make deposits.

Consequently there are two very different modes of transfer: by simple transfer of an asset as in FIG. 1a or by creation of a new A-L pair $\Delta$ as in FIG. 1b. In a monetary system that is not homogeneous in time, frequent creation and annihilation of monetary pairs leads to fluctuations in the money supply within the bounds set by the required reserve ratio.

As depicted in FIG. 1c the pair $\Delta$ created at $t^*$ is equivalent to unwarranted transfers

$N \to (i,j)$ of both assets and liabilities from all agents in the economy $N$ to the creation pair $(i,j)$. Another way to understand this process is shown in FIG. 2c. Given the total value $M = \sum |p_i|$ stored in the monetary memory, the relative wealth of all money holders is rescaled from $p_i/M$ to $p_i/(M+\Delta)$ and hence diminished by the white areas because each monetary unit represents a fraction of the total value stored. A higher amount of asset units in circulation therefore changes the unit of account. If payment is done by transfer of an already existing asset as in FIG. 1a, the relative wealth of agents not participating in the exchange remains unchanged.

The creation of money can therefore be understood either as a indirect transfer from creditors to debtors (FIG. 1c) or a rescaling equivalent to a change in the unit of account (FIG. 2c).

## I. A MONETARY NOETHER THEOREM

Bookkeeping is a memory of economic transfers, not a physical model of their dynamics. Physical laws of motion are symmetric by nature governed by the action principle. The properties of economic exchange influence but not determine the dynamics of the real economy. It is a matter of social choice whether the rules of exchange are invariant under certain transformations. Here we show that if those rules obey certain symmetries, these rules imply conservation laws. The stability of the unit of account and the elimination of non-local transfers is linked to time homogeneity by a monetary Noether theorem. Notably this result is independent of the existence of an action principle in bookkeeping.

*a. Momentum conservation* has a corresponding rule in bookkeeping[10], called the realization principle. It expresses value conservation by stating that revenue – a gain in assets or loss in liabilities – should only be recognized when an enforceable claim against another party exists to receive the revenue[9]. As shown below the realization principle is a consequence of symmetry under exchange of trading roles. Let

$$p_i^{(t)} = a_i^{(t)} - l_i^{(t)} \tag{1}$$

denote the account balance of agent $i$ before $(t = 1)$ and after $(t = 2)$ the transfer. The claim is that $p$ is conserved if the transfer is invariant under an exchange of transfer roles, that is, under index permutation $\pi_{ij}$. Let's write a transfer from $i \to j$ as

$$p_i^{(1)} - \Delta_i + p_j^{(1)} + \Delta_j = p_i^{(2)} + p_j^{(2)} \tag{2}$$

This does not say anything about conservation yet; it may well be that $\Delta_i \neq \Delta_j$ and hence $p_i^{(2)} + p_j^{(2)} \neq p_i^{(1)} + p_j^{(1)}$. Applying $\pi_{ij}$ to both sides gives $p_j^{(1)} - \Delta_j + p_i^{(1)} + \Delta_i = p_j^{(2)} + p_i^{(2)}$. The sum $p_j^{(2)} + p_i^{(2)}$ has obviously not changed; the left hand sides however are only equal if $\Delta_j = \Delta_i$ and hence only if momentum is conserved. Note that it is not necessary for either $i$ or $j$ to actually hold $\Delta$: as long as $\Delta_i = \Delta_j$, momentum is conserved. The creation of new A-L pairs is not prohibited.

*b. Energy conservation* has no corresponding principle in bookkeeping, which already indicates that time homogeneity is systematically violated in bookkeeping. As we saw in FIG. 1, this leads to problems when new monetary units are created.

Assume two agents hold $a_i(t) + a_j(t)$ at $t$. We shift the system – the holdings $a_i(t), l_i(t)$ – by $\delta t$: $a_i(t + \delta t) = a_i(t) + \partial_t a_i \delta t$ and $a_j(t + \delta t) = a_j(t) + \partial_t a_j \delta t$. If there is no transfer in $(t, t + \delta t)$, the derivatives are zero; otherwise

$$a_i(t) + a_j(t) = a_i(t) + \frac{\partial a_i}{\partial t}\delta t + a_j(t) + \frac{\partial a_j}{\partial t}\delta t \tag{3}$$

and time homogeneity $a_i(t) + a_j(t) = a_i(t + \delta t) + a_j(t + \delta t)$ is ensured if

$$\frac{\partial a_i}{\partial t} = -\frac{\partial a_j}{\partial t} \Rightarrow \frac{\partial}{\partial t}\sum_m a_m = 0 \tag{4}$$

which expresses conservation of asset units accross time and transfers. An equivalent expression can be derived for liability accounts $l_i$. If energy is not conserved, new money can be created during transfer. Due to momentum conservation it is created as a pair and we find $\partial_t a_i = \partial_t l_j$ as shown in FIG. 1b.

A corollary of EQ. 4 is that the payer needs to get hold of the assets before the transfer. Since $a_i(t) > 0 \ \forall i, t$, we have $a_i(t) > \Delta$ for all transfers $\Delta$ from $i \rightarrow j$.

Thus some transfers fail only because liquidity is lacking. A stricter order of transfers is enforced as liquidity has to be obtained from other agents before the transfer.

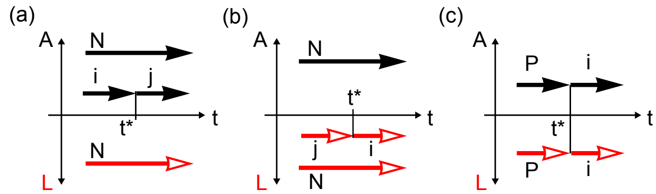


FIG. 3. Payment options in a symmetric monetary space. (a) Goods can be exchanged for assets or (b) passed along with liabilities. (c) In addition we allow liquidity transfers. All transfers preserve the total amount of monetary units in contrast to the pair creation in Fig. 1b.

## II. TIME SYMMETRIC MONEY

Above conservation principles can easily be interpreted as game theoretic concepts: momentum conservation corresponds to the zero sum principle, energy conservation implies a fixed pie game. Violation of energy conservation is a common feature of many economic standard models, which hand out new asset units each period as 'helicopter money'[7] and therefore are not homogeneous in time. In the same manner banks loan money into existence by creating A-L pairs. The charging of interest also creates additional A-L pairs[11] and is used to regulate the money supply indirectly by changing the price of credit over time[2]. Both creation processes are prohibited in a symmetric monetary system where the indirect transfers shown in Fig. 1b,c are prevented.

Without these creation processes a *symmetric monetary system* must contain an equal number $M$ of A- and L-units. Both units are freely transferable among agents using the time homogenous transfer types shown in Fig. 3. The money supply $M$ and account balances $p = a - l$ (Eq. 1) are conserved. Both A- and L-units are treated as a currency. The complementary units are represented in a central bank which monitors what the agents do. To each currency corresponds a price level; hence an agent's monetary wealth with holdings $(a, l)$ in a symmetric monetary space is given by

$$w = \frac{a}{p_a} - \frac{l}{p_l} \tag{5}$$

Nominal value $p$ is conserved whereas wealth $w$ is not. The two price levels $p_a$ for assets and $p_l$ for liabilities lead to a different scaling of the $y$-axis in Fig. 2 with an equilibrium exchange rate $e_{eq} = p_a/p_l$ between the two currencies.

Since the number of transactions in a random economy scales with the population and the average supply of money per agent $T = M/N$ is the scaling parameter in the distribution of money holdings, $T$ units of both types are issued or redeemed irrespective of the prevailing exchange rate when an agent enters or exits the system. The exit of an agent is equivalent to a credit default. On exit the last transfer partners act as creditors and debtors; they have to guarantee that the initially issued monetary units can be removed from the system. The risk of default will therefore either be included into prices or more likely lead to the establishment of an assurance infrastructure by banks similar to contemporary deposit insurance. These implications will be detailed in a subsequent manuscript.

## III. STATISTICAL MECHANICS OF SYMMETRIC MONEY

We simulated the effect of a symmetric money on the equilibrium distributions using random transfer[5] with a fixed number of agents $N$ and exponentially distributed transfer prices $x_i$, $i \in \{a, l\}$

$$\text{prob}(x_i) = \frac{1}{p_i}\mathrm{e}^{-x_i/p_i} \tag{6}$$

The possible transfers are depicted in FIG. 3. As shown in FIG. 4a, the distributions for assets $n(a)$ and liabilities $n(l)$ both equilibrate to Boltzmann-Gibbs distributions[6] with average money supply per person as temperature $T_{a,l} = M/N$:

$$n(a) = c_a \mathrm{e}^{-a/T_a} \qquad n(l) = c_l \mathrm{e}^{-l/T_l} \tag{7}$$

Empirically, monetary wealth in many societies follows this exponential distribution for all but the richest subpopulation[3]. All figures were generated with $M/N = p_a = p_l = 10$ unless indicated otherwise.

## IV. THE MARKET FOR LIQUIDITY

Liquidity shortage is the lack of means of payment for transaction purposes even though creditworthiness is ensured. In a non-symmetric monetary system liquidity is obtained by transferring monetary wealth (FIG. 1a) or taking out a loan (FIG. 1b). Creation of monetary units and liquidity necessarily occur at the same time. In contrast, FIG. 3c shows the transfer of liquidity from $P$ to $i$ without creation in a symmetric monetary system: when

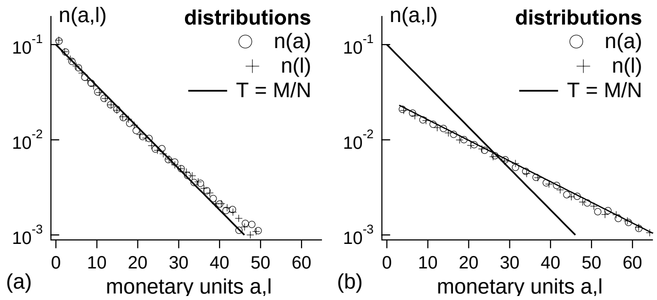


FIG. 4. Equilibrium distributions of symmetric money. (a) Without liquidity transfers (Fig. 3c), the system equilibrates to a Boltzmann-Gibbs distribution with $T = M/N$. (b) The liquidity market modifies the bath temperatures ($\phi = 1$).

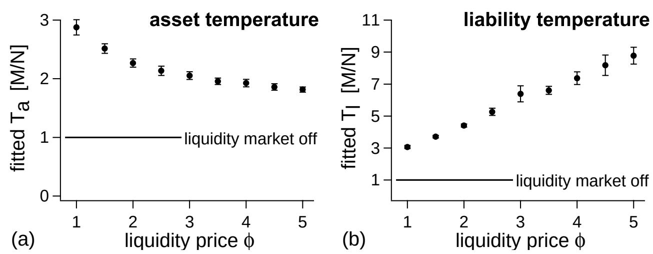


FIG. 5. The liquidity price $\phi$ changes the effective money supply per agent, $T$. The temperature of the equilibrium distribution shifts from the non-market value $M/N$ in (a) the asset bath (b) the liability bath.

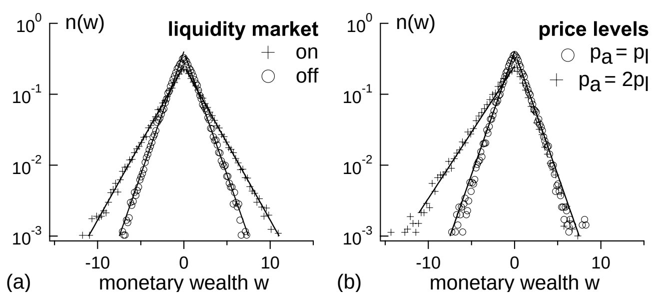


FIG. 6. (a) Effect of the liquidity market on the equilibrium distribution $n(w)$ of monetary wealth for $\phi = 1$ at $p_a = p_l = T = M/N$. (b) Effect of different price levels without a liquidity market. Both effects may superpose (not shown).

a providing agent $P$ transfers $\delta a$ A-units and $\delta l$ L-units simultaneously to another agent $i$, his wealth changes by $\Delta w_P = -\delta a/p_a + \delta l/p_l$. We call $\phi = \delta l/\delta a$ the price of liquidity; if it is equal to the inverse of the equilibrium exchange rate between A- and L-units $1/e_{eq}$, no wealth is transferred. However if $\phi > 1/e_{eq}$, a liquidity transfer of $a$ asset units and $a\phi$ liability units representing $\Delta w = a/p_a$ of purchasing power implies a profit for $P$ of

$$\Delta w_P(\phi) = (e_{eq}\phi - 1)\,\Delta w = k\Delta w \tag{8}$$

where $k$ plays the role of an interest rate. $\phi < 1/e_{eq}$ corresponds to a nominal interest rate $k < 0$, when lending money becomes unattractive. Note that liquidity is no longer lent, but bought on the liquidity market. The value of a transaction for $P$ depends ultimately on the *expected future* values of the A- and L-currencies.

The liquidity market significantly increases the effective money supply per agent $T$, as shown for $\phi = 1$ in FIG. 4b. The lower the price $\phi$ of liquidity, the more the asset temperature increases whereas the liability temperature decreases (FIG. 5). As expected, the liquidity market improves the efficient use of the nominal money supply: monetary units that are not part of a debt relation can be transferred to relax liquidity constraints of payers. The additional splits of liquidity into debt pairs lead to a higher temperature and debt level for any $\phi$; given exponential distributions of assets and liabilities however, the probability to find a liquidity provider decreases for higher $\phi$.

FIG. 6a shows the change in the wealth distribution $n(w)$ if a liquidity market is present. The changes in $n(w)$ induced by the liquidity market can be emulated by an increase in nominal money supply $T = M/N$ or by changes of price levels in EQ. 5. The effect of price level changes is shown in FIG. 6b, where the liquidity market was switched off.

## V. CONCLUSION

We found a monetary Noether theorem and showed that the symmetry properties of a monetary system have profound implications for its function as a memory of economic transactions. The common assumption of a fixed money supply is really an assumption about the time structure of the system. In a time-inhomogeneous monetary system we find non-local transfers between agents. These unwarranted transfers compromise the monetary memory (FIG. 1b,c and FIG. 2b,c). In a time-homogeneous monetary system these transfers are eliminated and replaced by local interactions on a liquidity market. This solves the problem of liquidity shortages under a fixed money supply.

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